A COMPARITIVE STUDY ON EIGHTH ORDER RATIONAL NUMERICAL INTEGRATOR WITH ADOMIAN DECOMPOSITION METHOD ON STIFF LINEAR SYSTEMS

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Abstract
In this paper, we compare the eighth order rational numerical integrator (RNI8) with Adomian decomposition method by solving a linear stiff differential system. This paper presents the numerical results to compare and show the performance of the methods using adaptive step-size control.

Keywords: stiff differential equations, rational numerical integrator, eighth order, adomian decomposition method.

1. INTRODUCTION

We consider the initial value problem (IVP)

$$y' = f(x, y),\ y(x_0) = y_0;\ y, f \in R^M \text{ and } x \in [a, b],\ a, b \in R \qquad (1)$$

whose solution may contain singularities. It is assumed that f(x, y) satisfy the Lipschitz condition.

Definition 1.1 Stiff IVP

A system of IVP of the form (1) is said to be stiff if the eigen value λ_i of the Jacobian matrix $\left[\frac{\partial f}{\partial y}\right]$ at every integration point x have negative real parts and differ greatly in magnitude.

Also, the eigen values λ_t satisfy the following conditions:

(i) $\mathrm{Re}(\lambda_t) < 0$, $t = 1,2, \ldots, m$ and

(ii) $\dfrac{\max_t |\lambda_t|}{\min_t |\lambda_t|} = S > 1$; S is the stiffness ratio.

Although the classical Runge-Kutta methods of higher order and implicit predictor-corrector methods are used for solving stiff equations they become very inefficient since the step size is controlled by stability requirement rather than accuracy requirement. They are based on the approximation of the solution by a polynomial, an approach that is too expensive when high accuracy is required [5-6].

Though most of the implicit methods work better in producing results for stiff problems than that of the explicit methods, some of the explicit rational non-linear schemes were proven to be efficient.The rational nonlinear schemes for the numerical solution of Eq.(1) are given in Lambert and Shaw [10], Luke et al [11] and Fatunla [7] ,Fatunla and Aashikpelokai [8], Ikhile [9], Otunta and Ikhile [12-13] and Otunta and Nwachukwu [15].Otunta and Nwachukwu [14] constructed schemes of order three, four, five, and six, seven respectively. Ponnammal and Prabu developed an eighth order rational numerical integrator [15] which is referred here as RNI8.

A power series method, called the Adomian decomposition method (ADM), can be used to derive an exact solution to a specific linear stiff system of IVPs [1-4]. The ADM gives an analytical solution in terms of a rapidly convergent infinite power series with easily calculatable terms. ADM and examples were discussed in [1-4].

The aim of this paper is to compare the numerical solutions of RNI8 [15] with ADM in a linear stiff system.

2 EIGHTH ORDER RATIONAL NUMERICAL INTEGRATOR (RNI8)

Otunta and Ikhile [14] considered the following rational function approximation for the IVP Eq.(1).

$$y(x) = A + \frac{xP_{k-1}(x)}{1+\sum_{j=1}^{K} b_j x^j}, \quad k > 0, \tag{2}$$

where $$P_k(x) = \sum_{j=0}^{k} a_j x^j, \quad k \geq 1 \tag{3}$$

We therefore consider the one-step scheme

$$y_{n+1} = A + \frac{x_{n+1}P_{k-1}(x_{n+1})}{1+\sum_{j=1}^{k} b_j x_{n+1}^j}, \quad k \geq 1 \tag{4}$$

Thus, we interpolate the theoretical solution of Eq. (1) by

$$y(x) = \frac{a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4}{1 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4} \tag{5}$$

The resultant one step scheme is given by

$$y_{n+1} = \frac{a_0 + a_1 x_{n+1} + a_2 x_{n+1}^2 + a_3 x_{n+1}^3 + a_4 x_{n+1}^4}{1 + b_1 x_{n+1} + b_2 x_{n+1}^2 + b_3 x_{n+1}^3 + b_4 x_{n+1}^4} \tag{6}$$

We write Eq.(6) as,

$$y_{n+1} = \left(a_0 + a_1 x_{n+1} + a_2 x_{n+1}^2 + a_3 x_{n+1}^3 + a_4 x_{n+1}^4\right)\left[1 + \sum_{r=1}^{\infty} (-1)^r \left(\sum_{j=1}^{4} b_j x_{n+1}^j\right)^r\right] \tag{7}$$

and superimposing it on

$$y(x_{n+1}) = \sum_{j=0}^{\infty} \frac{h^j y_n^{(j)}}{j!}; \qquad y_n^{(a)} = y_a, \tag{8}$$

we get,

$$\left(a_0 + a_1 x_{n+1} + a_2 x_{n+1}^2 + a_3 x_{n+1}^3 + a_4 x_{n+1}^4\right) = \left(1 + b_1 x_{n+1} + b_2 x_{n+1}^2 + b_3 x_{n+1}^3 + b_4 x_{n+1}^4\right)$$
$$\left(y_n + h y_n^{I} + \frac{h^2 y_n^{II}}{2!} + \frac{h^3 y_n^{III}}{3!} + \frac{h^4 y_n^{IV}}{4!} + \frac{h^5 y_n^{V}}{5!} + \frac{h^6 y_n^{VI}}{6!} + \frac{h^7 y_n^{VII}}{7!} + \frac{h^8 y_n^{VIII}}{8!}\right) + 0(h^9) \tag{9}$$

We obtain the method parameters from (9) as :

$$a_0 = y_n \quad (10)$$

$$a_1 = y_n b_1 + \frac{hy_n^I}{x_{n+1}} \quad (11)$$

$$a_2 = y_n b_2 + \frac{hy_n^I}{x_{n+1}} b_1 + \frac{h^2 y_n^{II}}{2!x_{n+1}^2} \quad (12)$$

$$a_3 = y_n b_3 + \frac{hy_n^I}{x_{n+1}} b_2 + \frac{h^2 y_n^{II}}{2!x_{n+1}^2} b_1 + \frac{h^3 y_n^{III}}{3!x_{n+1}^3} \quad (13)$$

$$a_4 = y_n b_4 + \frac{hy_n^I}{x_{n+1}} b_3 + \frac{h^2 y_n^{II}}{2!x_{n+1}^2} b_2 + \frac{h^3 y_n^{III}}{3!x_{n+1}^3} b_1 + \frac{h^4 y_n^{IV}}{4!x_{n+1}^4} \quad (14)$$

We arrive at a system of simultaneous equations from Eq.(10) – Eq.(14) where for each positive integer m, the term $\frac{h^m y_n^m}{m!x_{n+1}^m}$ is a real number. The system,in matrix form, is as shown below:

$$\begin{bmatrix} \frac{h^4 y_n^{IV}}{4!x_{n+1}^3} & \frac{h^3 y_n^{III}}{3!x_{n+1}^2} & \frac{h^2 y_n^{II}}{2!x_{n+1}} & hy_n^I \\ \frac{h^5 y_n^{V}}{5!x_{n+1}^3} & \frac{h^4 y_n^{IV}}{4!x_{n+1}^2} & \frac{h^3 y_n^{III}}{3!x_{n+1}} & \frac{h^2 y_n^{II}}{2!} \\ \frac{h^6 y_n^{VI}}{6!x_{n+1}^3} & \frac{h^5 y_n^{V}}{5!x_{n+1}^2} & \frac{h^4 y_n^{IV}}{4!x_{n+1}} & \frac{h^3 y_n^{III}}{3!} \\ \frac{h^7 y_n^{VII}}{7!x_{n+1}^3} & \frac{h^6 y_n^{VI}}{6!x_{n+1}^2} & \frac{h^5 y_n^{V}}{5!x_{n+1}} & \frac{h^4 y_n^{IV}}{4!} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} \frac{-h^5 y_n^{V}}{5!x_{n+1}^4} \\ \frac{-h^6 y_n^{VI}}{6!x_{n+1}^4} \\ \frac{-h^7 y_n^{VII}}{7!x_{n+1}^4} \\ \frac{-h^8 y_n^{VIII}}{8!x_{n+1}^4} \end{bmatrix} \quad (15)$$

This is of the form AX = B, solving this to obtain b_1,b_2,b_3,b_4 as in [15] so that we get,

$$b_1 = \frac{hV}{20Ux_{n+1}} \quad (16)$$

$$b_2 = \frac{h^2 S}{30Ux_{n+1}^2} \quad (17)$$

$$b_3 = \frac{h^3 W}{120Ux_{n+1}^3} \quad (18)$$

$$b_4 = \frac{h^4 R}{60Ux_{n+1}^4} \quad (19)$$

$$a_1 = \frac{h[Vy_n + 20Uy_n^I]}{20Ux_n} \quad (20)$$

$$a_2 = \frac{h^2[2Sy_n + 3Vy_n^I + 30Uy_n^{II}]}{60Ux_{n+1}^2} \quad (21)$$

$$a_3 = \frac{h^3[Wy_n+4Sy_n^I+3Vy_n^{II}+20y_n^{III}]}{120Ux_{n+1}^3} \tag{22}$$

$$a_4 = \frac{h^4[2Ry_n+Wy_n^I+2Sy_n^{II}+5Uy_n^{IV}]}{120Ux_{n+1}^4} \tag{23}$$

and

$$y_{n+1} = \frac{120Uy_n+6hA+2\;{}^2B+h^3C+Dh^4}{120U+6hV+4\;{}^2S+h^3W+2h^4R} \tag{24}$$

where

$A = Vy_n + 20Uy_n^I$
$B = 2Sy_n + 3Vy_n^I + 30Uy_n^{II}$
$C = Wy_n + 4Sy_n^I + 3Vy_n^{II} + 20y_n^{III}$
$D = 2Ry_n + Wy_n^I + 2Sy_n^{II} + Vy_n^{III} + 5Uy_n^{IV}$

and U,V,W,S,R are derived as in [15] and Eq.(24) gives the desired rational numerical integrator. The L-stability, consistence and convergence were proved for this method and the Local Truncation Error is obtained in [15].

3. ADOMIAN DECOMPOSITION METHOD

Let us consider the system of ordinary differential equation

$$y_i' = f_i(y_1, y_2, ..., y_n) + g_i, \qquad i = 1,2,...,m. \tag{25}$$

where f_i are nonlinear functions, g_i are known functions, and we are seeking the solution y_i satisfying (15). We assume that for every g_i, Eq. (25) has one and only one solution.

Applying the decomposition method as in [1-4], Eq. (25) can be written as

$$Ly_i = N_i(y_1, y_2, ..., y_n) + g_i, \qquad i = 1,2,...,m. \tag{26}$$

where $L = \frac{d}{dt}$ is the linear operator and $N_i(y_1, y_2, ..., y_n) = f_i(y_1, y_2, ... y_n)$ are the nonlinear operators. Operating on both sides of Eq. (26) with the inverse operator of L (namely $L^{-1}[.]dt$) gives

$$y_i = y_i(0) + L^{-1}N_i(y_i, y_2, ..., y_n) + L^{-1}g_i. \tag{27}$$

The Adomian technique consists of approximating the solution of Eq. (27) as an infinite series.

$$y_i = \sum_{n=0}^{\infty} y_i, n, \quad i = 1,2, ..., m \tag{28}$$

and decomposing the nonlinear operator N_i as

$$N_i(y_1, y_2, ..., y_n) = \sum_{n=0}^{\infty} A_i, n, \qquad i = 1,2, ..., m, \tag{29}$$

where $A_{i,n}$ are polynomials (called Adomian's polynomials) of y_0, ..., y_n [1-3] given by

$$A_{i,n} = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N_i\left(\sum_{i=0}^{\infty}\lambda^i y_i\right)\right]_{\lambda=0}, \qquad n = 0, 1, 2, ...$$

The series $\sum_{n=0}^{\infty} y_{i,n}$ and $\sum_{n=0}^{\infty} A_{i,n}$ are convergent as given in [1], Substituting Eq.(28) and Eq.(29) into Eq.(27) yields

$$\sum_{n=0}^{\infty} y_{i,n} = y_i(0) + L^{-1}\sum_{n=0}^{\infty} A_{i,n} + L^{-1}g_i.$$

Thus, we can arrive at the method

$$y_{i,0} = y_i(0) + L^{-1}g_i,$$
$$y_{i,n+1} = L^{-1}A_{i,n}(y_0,...,y_n), \qquad i = 1,2, ..., \qquad n = 0, 1, 2, ... \qquad (30)$$

Thus all components of y_i can be calculated once the $A_{i,n}$ are given. We then define the n-term approximant to the solution y_i by $\phi_{i,n}[y_i] = \sum_{k=0}^{n-1} y_{i,k}$ with $\lim_{n\to\infty} \phi_{i,n}[y_i] = y_i$.

The convergence is achieved by carrying out an efficient step-size control. A subinterval is determined where the condition $\|y_{i,n+1}\|_2 < x\|y_{i,n}\|_2$ holds for n = 0,1, ..., k, i = 1,2,..., m where 0< α<1 is a constant and k is the maximum order of the approximant used in the computation and m the number of equation [4].

4. NUMERICAL EXPERIMENT

Consider now the linear initial value problem [4]

$$\begin{aligned} y_1' &= -20y_1 - 0.25y_2 - 19.75y_3, & y_1(0) &= 1, \\ y_2' &= -20y_1 - 20.25y_2 + 0.25y_3, & y_2(0) &= 1, \\ y_3' &= -20y_1 - 19.75y_2 - 0.25y_3, & y_3(0) &= -1. \end{aligned} \qquad (31)$$

The exact solution of (31) is

$$y_{1E}(t) = \frac{1}{2}\left[e^{-0.5t} + e^{-20t}\left\{\cos(20t) + \sin(20t)\right\}\right],$$
$$y_{2E}(t) = \frac{1}{2}\left[e^{-0.5t} - e^{-20t}\left\{\cos(20t) - \sin(20t)\right\}\right],$$
$$y_{3E}(t) = -\frac{1}{2}\left[e^{-0.5t} + e^{-20t}\left\{\cos(20t) - \sin(20t)\right\}\right]$$

Appling the decomposition method as in [1-4], Eq. (31) can be written as

$$\begin{aligned} Ly_1 &= 20y_1 - 0.25y_2 - 19.75y_3, \\ Ly_2 &= 20y_1 - 20.25y_2 + 0.25y_3, \\ Ly_3 &= 20y_1 - 19.75y_2 - 0.25y_3, \end{aligned} \qquad (32)$$

where $L = \dfrac{d}{dt}$ is the linear operator. Operating on both sides of Eq. (32) with the inverse operator of L (namely $L^{-1}[.] = \int_0^t [.]\,dt\,dt$) gives

$$\begin{aligned} y_1 &= y_1(0) - 20L^{-1}y_1 - 0.25L^{-1}y_2 - 19.75L^{-1}y_3, \\ y_2 &= y_2(0) + 20L^{-1}y_1 - 20.25L^{-1}y_2 + 0.25L^{-1}y_3, \\ y_3 &= y_3(0) + 20L^{-1}y_1 - 19.75L^{-1}y_2 - 0.25L^{-1}y_3. \end{aligned} \qquad (33)$$

Therefore, the iterations are

$$y_{1,0}=1,\quad y_{2,0}=0,\quad y_{3,0}=-1,$$
$$y_{1,n+1}=-20L^{-1}y_{1,n}-0.25L^{-1}y_{2,n}-19.75L^{-1}y_{3,n},$$
$$y_{2,n+1}=20L^{-1}y_{1,n}-20.25L^{-1}y_{2,n}+0.25L^{-1}y_{3,n},$$
$$y_{3,n+1}=20L^{-1}y_{1,n}-19.75L^{-1}y_{2,n}-0.25L^{-1}y_{3,n}\ n=0,1,2...$$

which is the desired method.

The problem in Eq.(30) is solved using RNI8 method given by Eq.(24) and ADM given by Eq.(30). The problem has been integrated on the interval [0, 2] and the results using MATLAB are presented in Table 1 in the interval [1, 2] for adaptive step size *h*. The errors have been defined as the maximum of the absolute errors on the nodal points in the integration interval. It is observed from Table 1 that RNI8 acts as a better method in this linear example.

Table 1 Comparison of Absolute errors of RNI8 with Adomian decomposition of Example 1

t	|Error| in RNI8 y_1	|Error| in Adomian y_1	|Error| in RNI8 y_2	|Error| in Adomian y_2	|Error| in RNI8 y_3	|Error| in Adomian y_3
1	3.7918×10^{-11}	4.2895×10^{-11}	2.5021×10^{-11}	4.9101×10^{-11}	4.1681×10^{-10}	5.2013×10^{-9}
2	8.5813×10^{-10}	8.6142×10^{-10}	8.5211×10^{-10}	8.6191×10^{-10}	7.7110×10^{-10}	8.6005×10^{-10}

5. CONCLUSION

We have made a comparative study between RNI8 and ADM to solve a nonlinear stiff initial value system in this paper. It is observed from the numerical results in Table 1 that RNI8 is superior in this system in terms of accuracy.

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